

STRAIGHT LINES

1. DISTANCE FORMULA

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

2. SECTION FORMULA

The $P(x, y)$ divided the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$, then ;

$$x = \frac{mx_2 + nx_1}{m+n}; y = \frac{my_2 + ny_1}{m+n}$$



- (i) If m/n is positive, the division is internal, but if m/n is negative, the division is external.
- (ii) If P divides AB internally in the ratio $m:n$ & Q divides AB externally in the ratio $m:n$ then
 P & Q are said to be harmonic conjugate of each other w.r.t. AB .

Mathematically, $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP , AB & AQ are in H.P.

3. CENTROID, INCENTRE & EXCENTRE

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC , whose sides BC , CA , AB are of lengths a , b , c respectively, then the co-ordinates of the special points of triangle ABC are as follows :

$$\text{Centroid } G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

$$\text{Incentre } I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right) \text{ and}$$

Excentre (to A) I_1

$$\equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right) \text{ and so on.}$$



- (i) Incentre divides the angle bisectors in the ratio, $(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.
- (ii) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.
- (iii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio $2:1$.
- (iv) In an isosceles triangle G , O , I & C lie on the same line and in an equilateral triangle, all these four points coincide.

4. AREA OF TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC , then its area is equal to

$$\Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ provided the vertices are}$$

considered in the counter clockwise sense.

The above formula will give a $(-)$ ve area if the vertices (x_i, y_i) , $i = 1, 2, 3$ are placed in the clockwise sense.





Area of n-sided polygon formed by points

$(x_1, y_1); (x_2, y_2); \dots (x_n, y_n)$ is given by :

$$\frac{1}{2} \left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \dots + \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} \right)$$

5. SLOPE FORMULA

If θ is the angle at which a straight line is inclined to the positive direction of x-axis, and $0^\circ \leq \theta < 180^\circ$, $\theta \neq 90^\circ$, then the slope of the line, denoted by m , is defined by $m = \tan \theta$. If θ is 90° , m does not exist, but the line is parallel to the y-axis. If $\theta = 0$, then $m = 0$ and the line is parallel to the x-axis.

If $A(x_1, y_1)$ & $B(x_2, y_2)$, $x_1 \neq x_2$, are points on a straight line, then the slope m of the line is given by :

$$m = \left(\frac{y_1 - y_2}{x_1 - x_2} \right)$$

6. CONDITION OF COLLINEARITY OF THREE POINTS

Points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if :

$$(i) \quad m_{AB} = m_{BC} = m_{CA} \text{ i.e. } \left(\frac{y_1 - y_2}{x_1 - x_2} \right) = \left(\frac{y_2 - y_3}{x_2 - x_3} \right)$$

$$(ii) \quad \Delta ABC = 0 \text{ i.e. } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$(iii) \quad AC = AB + BC \text{ or } AB \sim BC$$

$$(iv) \quad A \text{ divides the line segment } BC \text{ in some ratio.}$$

7. EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

$$(i) \quad \text{Point-Slope form : } y - y_1 = m(x - x_1) \text{ is the equation of a straight line whose slope is } m \text{ and which passes through the point } (x_1, y_1).$$

$$(ii) \quad \text{Slope-Intercept form : } y = mx + c \text{ is the equation of a straight line whose slope is } m \text{ and which makes an intercept } c \text{ on the } y\text{-axis.}$$

$$(iii) \quad \text{Two point form : } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \text{ is the equation of a straight line which passes through the point } (x_1, y_1) \text{ \& } (x_2, y_2)$$

$$(iv) \quad \text{Determinant form : Equation of line passing through}$$

$$(x_1, y_1) \text{ and } (x_2, y_2) \text{ is } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$(v) \quad \text{Intercept form : } \frac{x}{a} + \frac{y}{b} = 1 \text{ is the equation of a straight line which makes intercepts } a \text{ \& } b \text{ on } OX \text{ \& } OY \text{ respectively.}$$

$$(vi) \quad \text{Perpendicular/Normal form : } x \cos \alpha + y \sin \alpha = p \text{ (where } p > 0, 0 \leq \alpha < 2\pi) \text{ is the equation of the straight line where the length of the perpendicular from the origin } O \text{ on the line is } p \text{ and this perpendicular makes an angle } \alpha \text{ with positive } x\text{-axis.}$$

$$(vii) \quad \text{Parametric form : } P(r) = (x, y) = (x_1 + r \cos \theta, y_1 + r \sin \theta) \text{ or}$$

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r \text{ is the equation of the line in parametric form, where 'r' is the parameter whose absolute value is the distance of any point } (x, y) \text{ on the line from fixed point } (x_1, y_1) \text{ on the line.}$$

$$(viii) \quad \text{General Form : } ax + by + c = 0 \text{ is the equation of a straight}$$

$$\text{line in the general form. In this case, slope of line} = -\frac{a}{b}.$$

8. POSITION OF THE POINT (x_1, y_1) RELATIVE OF THE LINE $ax + by + c = 0$

If $ax_1 + by_1 + c$ is of the same sign as c , then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c , the point (x_1, y_1) will lie on the non-origin side of $ax + by + c = 0$.

In general two points (x_1, y_1) and (x_2, y_2) will lie on same side or opposite side of $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.



9. THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS

Let the given line $ax + by + c = 0$ divides the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$, then

$$\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}. \text{ If A and B are on the same side of}$$

the given line then m/n is negative but if A and B are on opposite sides of the given line, then m/n is positive.

10. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of perpendicular from $P(x_1, y_1)$ on

$$ax + by + c = 0 \text{ is } \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

11. REFLECTION OF A POINT ABOUT A LINE

- (i) The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2 \frac{ax_1 + by_1 + c}{a^2 + b^2}$$

- (ii) Similarly foot of the perpendicular from a point on the line is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

12. ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES

If m_1 and m_2 are the slopes of two intersecting straight lines ($m_1 m_2 \neq -1$) and θ is the acute angle between them,

$$\text{then } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$



Let m_1, m_2, m_3 are the slopes of three line $L_1=0; L_2=0; L_3=0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these lines are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}; \tan B = \frac{m_2 - m_3}{1 + m_2 m_3}; \text{ and } \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

13. PARALLEL LINES

- (i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where d is parameter.
- (ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel

$$\text{if : } \frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$$

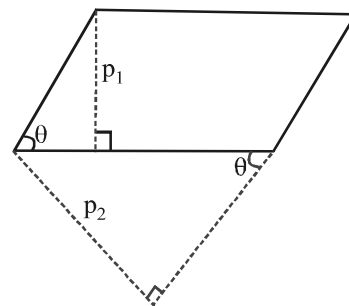
Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

- (iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ and

$$ax + by + c_2 = 0 \text{ is } \left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right|$$

Coefficient of x & y in both the equations must be same.

- (iv) The area of the parallelogram = $\frac{p_1 p_2}{\sin \theta}$, where p_1 and p_2 are



distance between two pairs of opposite sides and θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1 x + c_1$, $y = m_1 x + c_2$, $y = m_2 x + d_1$, $y = m_2 x + d_2$ is given by

$$\left| \frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2} \right|$$



14. PERPENDICULAR LINES

- (i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slope is -1 i.e., $m_1 m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form.

$$y = -\frac{1}{m}x + d, \text{ where } d \text{ is any parameter.}$$

- (ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

15. STRAIGHT LINES MAKING ANGLE α WITH GIVEN LINE

The equation of lines passing through point (x_1, y_1) and making angle α with the line $y = mx + c$ are given by $(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan \theta = m$.

16. BISECTOR OF THE ANGLES BETWEEN TWO LINES

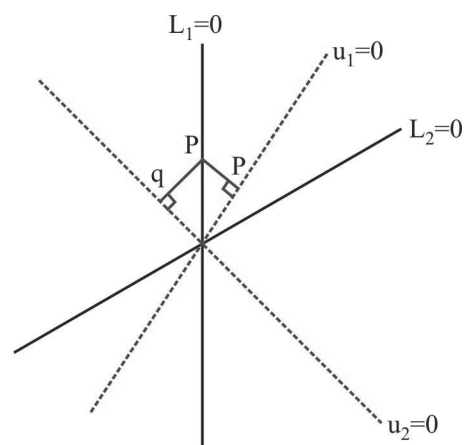
Equations of the bisectors of angles between the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ ($ab' \neq a'b$) are :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$



Equation of straight lines through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisector between these two lines & passing through the point P .

17. METHODS TO DISCRIMINATE BETWEEN THE ACUTE BISECTOR AND THE OBTUSE ANGLE BISECTOR



- (i) If θ be the angle between one of the lines & one of the bisectors, find $\tan \theta$. if $|\tan \theta| < 1$, then $2\theta < 90^\circ$ so that this bisector is the acute angle bisector. if $|\tan \theta| > 1$, then we get the bisector to be the obtuse angle bisector.
- (ii) Let $L_1=0$ & $L_2=0$ are the given lines & $u_1=0$ and $u_2=0$ are bisectors between $L_1=0$ and $L_2=0$. Take a point P on any one of the lines $L_1=0$ or $L_2=0$ and drop perpendicular on $u_1=0$ and $u_2=0$ as shown. If.

$$|p| < |q| \Rightarrow u_1 \text{ is the acute angle bisector.}$$

$$|p| > |q| \Rightarrow u_1 \text{ is the obtuse angle bisector.}$$

$$|p| = |q| \Rightarrow \text{the lines } L_1 \text{ and } L_2 \text{ are perpendicular.}$$

- (iii) if $aa' + bb' < 0$, while c & c' are positive, then the angle between the lines is acute and the equation of the bisector

$$\text{of this acute angle is } \frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

If, however, $aa' + bb' > 0$, while c and c' are positive, then the angle between the lines is obtuse & the equation of the bisector of this obtuse angle is :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

The other equation represents the obtuse angle bisector in both cases.



18. TO DISCRIMINATE BETWEEN THE BISECTOR OF THE ANGLE CONTAINING A POINT

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equation, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant term c, c' are positive.

Then ; $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of

the bisector of the angle containing origin and

$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the

bisector of the angle not containing the origin. In general equation of the bisector which contains the point (α, β) is.

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}} \text{ or } \frac{ax + by + c}{\sqrt{a^2 + b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

according as $a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

19. CONDITION OF CONCURRENCY

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Alternatively : If three constants A, B and C (not all zero) can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

20. FAMILY OF STRAIGHT LINES

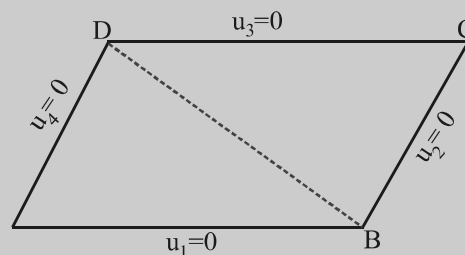
The equation of a family of straight lines passing through the points of intersection of the lines,

$L_1 \equiv a_1x + b_1y + c_1 = 0$ & $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + kL_2 = 0$ i.e.

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.



- (i) If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$, then $u_1 = 0$; $u_2 = 0$; $u_3 = 0$; $u_4 = 0$; form a parallelogram



The diagonal BD can be given by $u_2u_3 - u_1u_4 = 0$

Proof : Since it is the first degree equation in x & y , it is a straight line. Secondly point B satisfies $u_2 = 0$ and $u_1 = 0$ while point D satisfies $u_3 = 0$ and $u_4 = 0$. Hence the result. Similarly, the diagonal AC can be given by $u_1u_2 - u_3u_4 = 0$

- (ii) The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some real λ and μ .

[For getting the values of λ and μ compare the coefficients of x, y & the constant terms.]

21. A PAIR OF STRAIGHT LINES THROUGH ORIGIN

- (i) A homogeneous equation of degree two, " $ax^2 + 2hxy + by^2 = 0$ " always represents a pair of straight lines passing through the origin if :

- $h^2 > ab \Rightarrow$ lines are real and distinct.
- $h^2 = ab \Rightarrow$ lines are coincident.
- $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. $(0,0)$

- (ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$



- (iii) If θ is the acute angle between the pair of straight lines represented by,

$$ax^2 + 2hxy + by^2 = 0, \text{ then; } \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

- (iv) The condition that these lines are :

- (a) At right angles to each other is $a + b = 0$ i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$
- (b) Coincident is $h^2 = ab$.
- (c) Equally inclined to the axis of x is $h = 0$ i.e. coeff. of $xy = 0$.



A homogeneous equation of degree n represents n straight lines passing through origin.

- (v) The equation to the pair of straight lines bisecting the angle between the straight lines,

$$ax^2 + 2hxy + by^2 = 0, \text{ is } \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

22. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES

- (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

- (ii) The angle θ between the two lines representing by a general equation is the same as that between two lines represented by its homogenous part only.

23. HOMOGENIZATION

The equation of a pair of straight lines joining origin to the points of intersection of the line

$L \equiv lx + my + n = 0$ and a second degree curve,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx \left(\frac{lx + my}{-n} \right) +$$

$$2fy \left(\frac{lx + my}{-n} \right) + c \left(\frac{lx + my}{-n} \right)^2 = 0$$

The equal is obtained by homogenizing the equation of curve with the help of equation of line.



Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$ where λ and μ are parameters.

